PARTIAL DIFFERENTIAL EQUATIONS: PART-3

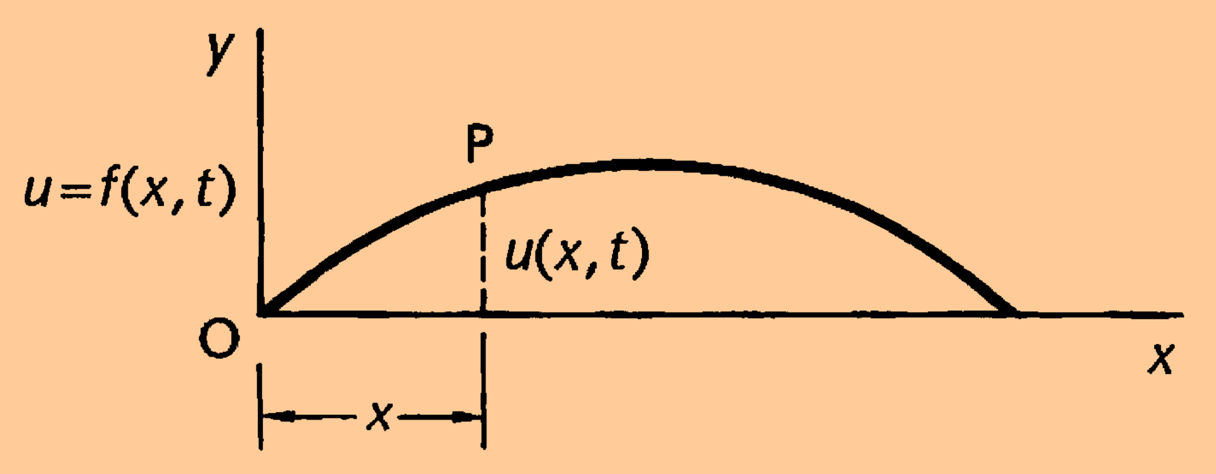
One-dimensional Heat Equation

Lecture #15

One-dimensional heat equation

The conduction of heat in a uniform bar depends on the initial distribution of temperature and on the physical properties of the bar, i.e. the thermal conductivity and specific heat of the material, and the mass per unit length of the bar.

With a uniform bar insulated except at its ends, any heat flow is along the bar and, at any instant, the temperature u at a point P is a function of its distance x from one end and of the time t.



One-dimensional heat equation

The one-dimensional heat equation is then of the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial u}{\partial t}$$

where $c^2 = \dfrac{k}{\sigma\rho}$ in which $k =$ thermal conductivity of the material; $\sigma =$ specific heat of the material; $\rho =$ mass per unit length of the bar.

You will already have noticed that the heat equation differs from the wave equation only in the fact that the right-hand side contains the first partial derivative instead of the second. It is not surprising therefore that the method of solution is very much like that of our previous examples.

Solution of one-dimensional heat equation

SUMMARY

$$\frac{d^2u}{dx^2} = \frac{1}{c^2} \cdot \frac{du}{dt}$$

The solution is:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

That satisfy the initial and boundary conditions:

$$u(0,t) = 0$$

$$u(L,t) = 0$$

$$u(x,0) = f(x) \quad \text{initial temperature}$$

$$\Rightarrow \quad u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x)$$

One-dimensional heat equation

Examples 1-2

1. Find the temperature $u(x,t)$ in a copper bar 80cm long if the initial temperature is $100\sin\frac{\pi x}{80}\,{}^{o}C$ and the ends are kept at $0^{o}C$. How long it will take for the maximum temperature in bar to drop to $50^{o}C$? ($c^2=1.158\ cm^2/s$)

2. Solve the problem in Example (1) above when the initial temperature is: $100\sin\frac{3\pi x}{80}\,{}^{o}C$ ← Sinusoidal initial temperature

GOOD LUCK FOR YOUR EXAM!!

BE SMART ☺

3. Find x.